

Perfect Graeco-Latin balanced incomplete block designs and related designs

Sunanda Bagchi

Theoretical Statistics and Mathematics Unit
Indian Statistical Institute
Bangalore 560059, India.

Abstract : Main effect plans orthogonal through the block factor (POTB) have been defined and a few series of them have been constructed in Bagchi (2010). These plans are very closely related to the ‘mutually orthogonal balanced nested row-column designs’ of Morgan and Uddin (1996) and many other combinatorial designs in the literature with different names like ‘BIBDs for two sets of treatment’, ‘Graeco-Latin designs’ and ‘PERGOLAs’. In fact all of them may be viewed as POTBs satisfying one or more additional conditions, making them ‘optimal’. However, the PERGOLAs are defined to satisfy an additional property, without which also it is optimal. Interestingly, this additional property is satisfied by all the hitherto known examples of POTBs, even when their definitions do not demand it.

In this paper we present direct and recursive constructions of POTBs. In the process we have constructed one design which seems to be the first example of an ‘optimal’ two-factor POTB which is not a PERGOLA (see Theorem 3.1).

1 Introduction.

Preece (1966) constructed ‘BIBDs for two sets of treatments’. Subsequently several authors constructed similar combinatorial objects. Among these, the ones relevant to the present paper are ‘balanced Graeco-Latin block designs’ of Seberry (1979), ‘Graeco-Latin designs of type 1’ of Street (1981) and ‘Perfect Graeco-Latin balanced incomplete block designs (PERGOLAs)’ of Rees and Preece (1999).

Morgan and Uddin (1996) considered main effects plans (MEPs) on a nested row-column set up and proved the optimality of ‘mutually orthogonal balanced nested row-column designs’. They also discussed the constructional aspects of such designs. Unfortunately, the relevance of these results in the context of blocked MEPs was overlooked by later authors studying blocked MEPs like Mukerjee, Dey and Chatterjee (2001) and Bagchi (2010). Optimal Main effects plans for three or more factors on non-orthogonal blocks of small size were obtained in Mukerjee, Dey and Chatterjee (2001). Bagchi (2010) defined and studied main effect plans orthogonal ‘through the block factor’ (POTB).

In the present paper we first note the relation between POTBs with optimality property (termed balanced POTB) and the combinatorial objects considered by earlier authors mentioned above. Next we construct a few series of POTBs. We also present a recursive construction by which the number of factors is multiplied, keeping the block size unchanged, thus yielding multi-factor POTBs from PERGOLAs and other similar two-factor designs.

We note that a PERGOLA is a balanced POTB with an additional condition. It is interesting that all the balanced POTBs available in the literature (with different names) do satisfy this condition. (Table 1 of Rees and Preece (1999) shows that such designs are plentiful). One would, therefore, suspect that this condition is implicit in the definition. We have, however, found a balanced POTB which does not satisfy this condition. [See Theorem 3.1].

2 Preliminaries

In this section we present the definition of a balanced POTB. We also list related combinatorial objects existing in the literature with various names.

Definition 2.1 By a **block design** with v treatments and b blocks of size k each we mean an **incidence structure** represented by a $v \times b$ matrix N having constant column sum k .

With any such block design D one associates the graph $G(D)$ with the treatments of D as vertices, two treatments being adjacent in $G(D)$ if there is a block containing both the treatments. One says that D is **connected** if the graph $G(D)$ is connected in the usual sense.

$\mathcal{D}(b, k, v)$ will denote the class of all connected block designs with v treatments on b blocks of size k each.

Definition 2.2 Consider an $(m+1) \times n$ array A such that the entries of the i th row are elements of a set S_i of size s_i , $i = 0, 1, \dots, m$. This is said to be a **main effect plan (MEP)** for $m+1$ factors, say, F_0, F_1, \dots, F_m on n runs. The i th row corresponds to the factor F_i and we say that F_i has s_i levels.

For $0 \leq i, j \leq m$, let M_{ij} be the $s_i \times s_j$ matrix such that the rows and columns of M_{ij} are indexed by S_i and S_j respectively and the (p, q) th entry of M_{ij} is the number of columns of A in which the i th and j th entries are p and q respectively, $p \in S_i$, $q \in S_j$. M_{ij} is said to be the F_i versus F_j **incidence matrix**.

Now, suppose $n = bk$, where b and k are integers. A **blocked MEP** (laid out on b blocks of size k each) is an $(m+1) \times bk$ array in which S_0 is the set of integers $\{1, 2, \dots, b\}$ and each integer $j \in S_0$ appears exactly k times in the last row of A . The 0-th row is said to correspond to the “**block factor**”, which is represented by B (and not F_0). In this case, the incidence matrix of the i th row versus the last row is denoted by M_{iB} , $1 \leq i \leq m$. A blocked MEP is said to be ‘**connected**’ (borrowing a term from the theory of block designs) if each M_{iB} is the incidence matrix of a connected block design. It is said to be **symmetric** if $s_1 = s_2 = \dots = s_m$. A symmetric MEP with $s_i = s$ for every i is also referred to as an **MEP for an s^m experiment**.

In the applications, there are n experimental units, which are classified into homogeneous classes or blocks. These units are used to study the effects of m factors, the i th one having s_i ‘levels’. Typically, an experimental unit, say in the j th block, receives a ‘level combination’ say $x = (x_1, \dots, x_m)$, i.e. the level x_i of F_i is applied on that unit, $i = 1, 2, \dots, m$. This information is stored in the column vector $(j, x_1, \dots, x_m)'$. The array A consists of all such column vectors.

Definition 2.3 [Bagchi(2010)] The i th and j th factors of a blocked MEP ρ are said to be **orthogonal through the block factor (OTB)** if

$$M_{iB}(M_{jB})' = kM_{ij}. \quad (2.1)$$

If every pair of factors of a plan ρ is orthogonal to each other through the block factor, then ρ is said to be a **plan orthogonal through the block factor (POTB)**.

Remark 2.1: What is the utility of condition (2.1) ? This condition guaranties that for the inference on a factor F_i of a POTB one has to look at only its incidence with the block factor (i.e. M_{iB}) and forget all other treatment factors. Thus, the performance of a POTB ρ regarding the inference on the i th treatment factor depends only on the incidence matrix M_{iB} . We present a more precise statement in the next theorem. We omit the proof which can be obtained by going along the same lines as in the proofs of Lemma 1 and Theorem 1 of Mukerjee, Dey and Chatterjee (2001). [See Shah and Sinha (1989) for definitions, results and other details about optimality].

Theorem 2.1 Suppose a connected POTB ρ^* satisfies the following condition. For some non-increasing optimality criterion ϕ , M_{iB} is the incidence matrix of a block design d which is ϕ -optimal in the class of all connected block designs with s_i treatments and b blocks of size k each. Then, ρ^* is ϕ -optimal in the class of all connected m -factor MEP in the same set-up for the inference on the i th factor..

In particular, using the well-known optimality property of a BIBD we get the following result.

Corollary 2.1 Suppose ρ^* is a connected POTB. Suppose further that M_{iB} is the incidence matrix of a BIBD. Then, ρ^* is universally optimal in the class of all m -factor connected MEP in the same set up, for the inference on the i th factor.

In view of the above result, we introduce the following term.

Definition 2.4 A connected POTB is said to be **balanced** if each of its factors form a BIBD with the block factor, that is M_{iB} is the incidence matrix of a BIBD for each factor F_i .

We now present a small example of a balanced POTB. This has two factors, each with four levels 0,1,2,3 on six blocks of size two each.

Blocks	B_1		B_2		B_3		B_4		B_5		B_6	
F_1	0	2	1	3	0	3	1	2	0	1	3	2
F_2	1	3	0	2	2	1	3	0	3	2	0	1

Next we list a few combinatorial designs and note their relation with balanced POTBs.

(a) **Balanced Graco-Latin block design** defined and constructed in Seberry (1979) are balanced POTBs with two factors.

(b) **Graco-Latin block design of type 1** of Street (1981) are also two-factor balanced POTBs having

$$\mathbf{M}_{12} = J.$$

(c) **Perfect Graeco-Latin balanced incomplete block designs (PERGOLAs)** defined and discussed extensively in Rees and Preece (1999) are two-factor balanced POTBs having

(i) $s_1 = s_2 = s$, say and

$$\mathbf{M}_{12}\mathbf{M}'_{12} = \mathbf{M}'_{12}\mathbf{M}_{12} = fI_s + gJ_s, \quad f, g \text{ are integers.} \quad (2.2)$$

(d) **Mutually orthogonal BIBDs** defined and constructed by Morgan and Uddin (1996) are multi-factor balanced POTBs.

Here I_n is the identity matrix and J_n is the all-one matrix of order n .

Remark 2.2: The definition of neither balanced Graco-Latin block designs nor of mutually orthogonal BIBDs include condition (2.2). However, it is interesting to note that all these designs constructed so far do satisfy this condition. [See theorem 3.5].

3 Constructions for symmetric POTBs

Now we present a few constructions of POTB's. Each of these constructions is in terms of some group G of order g (which is the additive group of the field V in Theorem 3.4).

A block will consist of k plots or runs represented by columns. By adding an element $u \in G$ to a block we mean adding u to the level of every factor in every run of the block. By developing an initial block we mean generating g blocks by adding distinct elements of G to the initial block.

Let N denote the set of integers modulo n and N^+ denote $N \cup \{\infty\}$.

Theorem 3.1 Let n be a positive integer ≥ 5 .

(a) Then there exists a POTB with three factors F_0, F_1, F_2 each having $n+1$ levels on $b = 6n$ blocks of size two.

(b) In the case $n = 5$, we get a Balanced POTB, which is **not a PERGOLA**.

Proof : (a) Let N^+ be the set of levels for each factor. The initial blocks B_{ij} , $i = 1, 2$, $j = 0, 1, 2$ are as follows. Here addition in the suffix of F is modulo 3.

	Block B_{1j}		Block B_{2j}		
F_{0+j}	∞	0	∞	0	, $j = 0, 1, 2$.
F_{1+j}	0	1	0	2	
F_{2+j}	-1	1	1	2	

That the design satisfies the required property follows by straightforward verification.

(b) Let $n = 5$. One can verify that the incidence matrices satisfy the following.

$$M_{ij} = \begin{bmatrix} 0 & 2 & 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 1 & 1 & 2 \\ 2 & 2 & 2 & 2 & 1 & 1 \\ 2 & 1 & 2 & 2 & 2 & 1 \\ 2 & 1 & 1 & 2 & 2 & 2 \\ 2 & 2 & 1 & 1 & 2 & 2 \end{bmatrix}, \quad i, j = 0, 1, 2 \text{ and} \quad (3.3)$$

$$M_{iB}(M_{iB})' = 8I_6 + 2J_6, \quad i = 1, 2, 3. \quad (3.4)$$

We see that each M_{iB} is the incidence matrix of a BIBD. Thus, by definition 2.4 it is a balanced POTB. But neither of M_{ij} s satisfy (2.2), as is clear from (3.3). Thus, the two-factor balanced POTB obtained by ignoring any one of the factors is not a PERGOLA. \square

Theorem 3.2 Suppose n is an integer ≥ 5 . Then there exists

(a) a POTB with two n -level factors on $2n$ blocks and

(b) a POTB with four n -level factors on $4n$ blocks of size 2 each.

(c) We get a balanced POTB in the case $n = 5$ in series (a). Further, in the case $n = 10$ in series (b) we get a POTB which is E -optimal for the inference on each factor.

Proof : The set of levels of each factor is N . Let $a, b \in N$.

(a) We present initial blocks B_1 and B_2 below.

	Block B_1		Block B_2	
F_1	a	$-a$	b	$-b$
F_2	b	$-b$	$-a$	a

(b) We present the initial blocks B_l , $l = 1, \dots, 4$ below.

Blocks	B_1		B_2		B_3		B_4	
F_1	0	a	a	$-a$	0	b	$-b$	b
F_2	a	$-a$	0	$-a$	$-b$	b	0	b
F_3	0	b	b	$-b$	$-a$	0	a	$-a$
F_4	b	$-b$	0	$-b$	a	$-a$	a	0

That these initial blocks generate POTBs can be verified by straightforward computation.

(c) For $n = 5$, taking $a = 1, b = 2$ we get a balanced POTB.

For $n = 10$, we take $a = 1$ and $b = 3$. Then for every $i = 1, \dots, 4$, M_{iB} is the incidence matrix of a group divisible design with five groups, the j th group being the pair of levels $\{j, j+5\}$ $j = 0, \dots, 4$, satisfying

$\lambda_0 = 0$ and $\lambda_1 = 1$. This plan is, therefore, E-optimal for the inference on all the four factors by Takeuchi (1961). \square

Theorem 3.3 (a) *There exists a symmetric POTB with four n -level factors on $4n$ blocks of size 2 each, whenever $n \geq 9$. We get a balanced POTB in the case $n = 9$.*

(b) *There exists a symmetric POTB with four factors each having $n + 1$ levels on $6n$ blocks of size 2 each, whenever $n \geq 7$.*

Proof : (a) The set of levels for each factor is N . Let $a, b, c, d \in N$. The initial blocks B_l , $l = 1, \dots, 4$ are as follows.

Blocks	B_1		B_2		B_3		B_4	
F_1	a	$-a$	b	$-b$	c	$-c$	$-d$	d
F_2	b	$-b$	$-a$	a	$-d$	d	$-c$	c
F_3	c	$-c$	d	$-d$	$-a$	a	b	$-b$
F_4	d	$-d$	$-c$	c	b	$-b$	a	$-a$

One can easily verify that these initial blocks generate a symmetric POTB with the given parameters. By taking $a = 1, b = 2, c = 3$ and $d = 4$ in the case $n = 9$, we get a balanced POTB.

(b) The set of levels for each factor is N^+ . Let $a, b, c \in N$. The initial blocks B_l , $l = 1, \dots, 6$ are as follows.

Blocks	B_1		B_2		B_3		B_4		B_5		B_6	
F_1	0	∞	a	$-a$	b	$-b$	c	$-c$	a	$-a$	a	$-a$
F_2	a	$-a$	0	∞	c	$-c$	$-b$	b	a	$-a$	$-a$	a
F_3	b	$-b$	c	$-c$	0	∞	a	$-a$	$-c$	c	$-c$	c
F_4	c	$-c$	b	$-b$	a	$-a$	0	∞	$-c$	c	c	$-c$

That the design satisfies the required property follows by straightforward verification. \square

Next we construct a series of balanced POTBs using finite fields. We first introduce the following notation.

Notation 3.1 (i) Let \sqcup denote an union counting multiplicity.

(ii) For a set A and an integer n , let nA denotes a multiset in which every member of A occurs n times.

(iii) For subsets A and B of a group $(G, +)$,

$$A - B = \{a - b : a \in A, b \in B\}.$$

Notation 3.2 (i) v denotes an odd prime or a prime power, written as $v = mf + 1$. V denotes the Galois field of order v . Further, $V^* = V \setminus \{0\}$ and $V^+ = V \cup \{\infty\}$.

(ii) α denotes a primitive element of V .

(iii) $\beta = \alpha^m$ is a generator of the subgroup C_0 of order f of (V^*, \cdot) .

(iv) C_0, C_1, \dots, C_{m-1} are the cosets of C_0 in (V^*, \cdot) .

(v) $(i, j) =$ the number of ordered pairs of integers (s, t) such that the following equation is satisfied in V . [This notation is borrowed from the theory of cyclotomy]

$$1 + \alpha^s = \alpha^t, \quad s \equiv i, t \equiv j \pmod{m}.$$

We need the following well-known result. [See Hall (1986), for instance].

Lemma 3.1 Suppose $m = 2$. Then the following hold.

(a) $-1 \in C_0$ (respectively C_1) if f is even (respectively odd).

(b) If f is even, then $\alpha - 1 \in C_i \Rightarrow \alpha^{-1} - 1 \in C_{i+1}$.

(c) If f is odd, then $\alpha - 1 \in C_i \Rightarrow 1 - \alpha^{-1} \in C_{i+1}$.

Here $+$ in the suffix is modulo 2.

We present the following well-known results for ready reference. [See equations (11.6.30), (11.6.40) and (11.6.43) of Hall (1986)].

Lemma 3.2 The differences between the cosets C_j 's of V^* can be expressed in terms of the cyclotomy numbers (p, q) 's as follows.

$$C_i - C_j = \begin{cases} \bigcup_{k=0}^{m-1} (k - j, i - j)C_k & \text{if } j \neq i \\ f\{0\} \cup \bigcup_{k=0}^{m-1} (k - j, 0)C_k & \text{if } j = i \end{cases}$$

The following cyclotomy numbers are known for the case $m = 2$.

Case 1: f odd. $(0, 0) = (1, 1) = (1, 0) = (f-1)/2$, $(0, 1) = (f+1)/2$.

Case 2: f even. $(0, 0) = f/2 - 1$, $(0, 1) = (1, 0) = (1, 1) = f/2$.

A series of two-factor balanced POTBs :

Theorem 3.4 Suppose v is an odd prime or a prime power. Then there exists a balanced POTB for a $(v + 1)^2$ experiment on $b = 2v$ blocks of size $(v + 1)/2$.

Proof : We write $v = 2f + 1$. The set of levels of each factor is $V \cup \{\infty\}$. The plan is obtained by developing the following initial blocks B_0 and B_1 presented below.

Case 1 : f is even.

$$B_0 = \begin{bmatrix} \infty & 1 & \beta & \dots & \beta^{f-1} \\ 0 & \alpha & \alpha\beta & \dots & \alpha\beta^{f-1} \end{bmatrix} \text{ and } B_1 = \begin{bmatrix} 0 & 1 & \beta & \dots & \beta^{f-1} \\ \infty & \alpha^{-1} & \alpha^{-1}\beta & \dots & \alpha^{-1}\beta^{f-1} \end{bmatrix}.$$

Case 2 : f is odd.

Block B_0 is as in case 1, while Block B_1 is as follows.

$$B_1 = \begin{bmatrix} 0 & \alpha^{-1} & \alpha^{-1}\beta & \dots & \alpha^{-1}\beta^{f-1} \\ \infty & 1 & \beta & \dots & \beta^{f-1} \end{bmatrix}.$$

Clearly block size is $f + 1 = (v + 1)/2$. To show that the plan satisfies the required property, we have to show that

(a) the plan is a POTB and (b) each factor forms a BIBD with the block factor.

Condition (b) follows from Lemma 3.2. So, we prove (a).

let us use the following **simplified notation** $\mathbf{M} = ((m_{ij}))$ for \mathbf{M}_{12} and \mathbf{A} for $\mathbf{M}_{1B}(\mathbf{M}_{2B})'$. We note that m_{ij} is the total number of plots (runs) in which the level combination (i, j) appears, while a_{ij} is the number of blocks in which F_1 is at level i and F_2 at level j , (in same or different plots).

We shall show that

$$M = J - I \text{ and} \tag{3.5}$$

$$A = (f + 1)(J - I) \tag{3.6}$$

We begin with M . It is clear that $m_{ii} = 0, i \in V^+$ and $m_{\infty, i} = m_{i, \infty} = 1, i \in V$.

We, therefore, assume $i \neq j$, $i, j \in V$. Let $u = j - i$. Then, m_{ij} is the number of times u appears in the multiset

$$\begin{cases} (\alpha - 1)C_0 \sqcup (\alpha^{-1} - 1)C_0 & \text{if } f \text{ is even} \\ (\alpha - 1)C_0 \sqcup (1 - \alpha^{-1})C_0 & \text{if } f \text{ is odd} \end{cases}$$

The relations above imply (3.5) in view of Lemma 3.1.

Now we look at A . Clearly, $A_{ii} = 0$, $i \in V^+$. Further, for every $i \in V$, $m_{\infty, i}$ is the replication number of i in the block design generated by the initial block $\{0\} \cup C_1$. Similarly, $m_{i, \infty}$ is the replication number of i in the block design generated by the initial block $\{0\} \cup C_0$ if f is odd and $\{0\} \cup C_1$ otherwise. Thus,

$$m_{\infty, i} = m_{i, \infty} = f + 1, i \in V.$$

Next we consider $i \neq j$, $i, j \in V$. Let $u = j - i$. Then, a_{ij} is the number of times u appears in the multiset

$$\tilde{S} = \begin{cases} ((\{0\} \cup C_1) - C_0) \sqcup (C_1 - (\{0\} \cup C_0)) & \text{if } f \text{ is even} \\ ((\{0\} \cup C_1) - C_0) \sqcup (C_0 - (\{0\} \cup C_1)) & \text{if } f \text{ is odd.} \end{cases}$$

These, together with Lemma 3.2 and (a) of Lemma 3.1 imply the equation next to (3.5). \square

Now we present the series of balanced POTBs available in the literature, together with two newly constructed balanced POTBs.

Notation 3.3 v denotes an odd prime or a prime power of the form $v = mf + 1$.

Table 3.1 : Balanced POTBs

No.	The expt.	m	# of Blocks	Block size	Inc. matrix (M_{ij})	Reference
1.	$(v + 1) \times v$	2	$2v$	$f = (v - 1)/2$	J	Seberry (1979)
2.	$(v + 1)^2$	2	$2v$	$f + 1 = (v + 1)/2$	J	Street (1981)
3(a).	v^m	m	mv	f	$(J - I)$	Morgan and Uddin(1996)
3(b).	v^t	tg	mv	$hf, h \leq g$	$h(J - I)$	Morgan and Uddin(1996)
4.	v^f	m	mv	$1 + hf, h \leq m$	$(m - h)I + hJ$	Morgan and Uddin(1996)
5.	$(v + 1)^2$	2	$2v$	$f + 1$	$J - I$	Theorem 3.4
6.	$(5 + 1)^3$	-	30	2	given in (3.3)	Theorem 3.1
7.	9^4	-	36	2	$J - I$	Theorem 3.3

In 3(b) above t is a factor of $m, g = m/t$ and h is an integer $\leq g$.

Using the information in Table 3.1, one may verify the following result.

Theorem 3.5 Every two-factor balanced POTB obtained from an existing multi-factor balanced POTB, except the one with $n = 5$, constructed in Theorem 3.1, is a PERGOLA.

One may also look at Table 1 of Rees and Preece (1999) for many more examples of PERGOLAS.

A recursive construction

Notation 3.4 An orthogonal array with m rows, n columns, k symbols and strength 2, will be denoted by $OA(n, m, k, 2)$.

Theorem 3.6 Suppose there exists a balanced POTB with f factors on b blocks of size k each, with $f \leq k$. If further an $OA(k^2, m + 1, k, 2)$ exists, then a balanced POTB with mf factors on bk blocks of size k each also exists.

The proof of this theorem is based on the following lemma.

Lemma 3.3 *Consider a set of k runs of a plan for an experiment with $f(\leq k)$ factors, such that no level of any factor is repeated. If an $OA(k^2, m+1, k, 2)$ exists, then there exists an MEP with f classes of m k -level factors on k blocks of size k each with the following property. Every factor is orthogonal (w.r.t. the block factor) to every factor of a different class.*

Proof : Let D denote the given set of runs. Let $F = \{P, Q, \dots\}$ denote the set of f factors of D . For each $P \in F$, let K_P denote the set of levels of P appearing in D . Let $K = \{1, \dots, k\}$ denote the set of symbols of the given OA (\tilde{O}), say. For every $P \in F$, let L_P denote the following one-one function from K to K_P . [By assumption, size of K_P is k for each P].

$$L_P(i) = j, i \in K, j \in K_P, \text{ if } P \text{ has level } j \text{ in the } i\text{th run of } D. \quad (3.7)$$

Let us arrange the columns of the given OA (\tilde{O}) as

$$(\tilde{O}) = [\tilde{A}_1 \quad \dots \quad \tilde{A}_k],$$

such that the 1st row of \tilde{A}_i consists of the symbol i repeated k times, $1 \leq i \leq k$. Let A_i denote the $m \times k$ array obtained from \tilde{A}_i by deleting the 1st row. Thus, every member of K appear exactly once in every row of each $A_i, i = 1, \dots, k$.

We now construct D^* , the reqd MEP. For each factor P of D , there will be m factors P_1, \dots, P_m in D^* , each of which will have K_P as the set of levels.

For $i \in K$, let us fix A_i and a factor, say P of D . If the j th column of A_i is $(s_1, \dots, s_m), s_u \in K$, then in the j -th plot of the i -th block of D^* , the factor P_t will have level $L_P(s_t), t = 1, 2, \dots, m$, where L_P is as in (3.7). Doing the same for all the factors and varying j over $\{1, 2, \dots, k\}$ we get a block of D^* . Finally varying i over K we generate the k blocks of the reqd MEP.

We now show that the MEP D^* satisfies the required property. We fix two factors, say P_i and $Q_j, i \neq j$ and an ordered pair of levels, say $(u, v), u \in K_P, v \in K_Q$. From the construction the following is clear. In every block there is a plot in which P_i is at level u and a plot where Q_j is at level v . Moreover, there is exactly one block in which these factors are set at these levels in the same plot. Thus, the factors P_i and Q_j are mutually orthogonal through the block factor. We see that if $P = Q$, then also the argument above holds. Thus P_i is orthogonal to $P_j, j \neq i$. However, P_i may not be orthogonal to Q_i . We, therefore form the classes as $C_i = \{P_i, Q_i, \dots\}, P, Q \in F, i = 1, 2, \dots, m$. Now the factors satisfy the orthogonality condition of the hypothesis. \square

Proof of the theorem : Let D denote the given POTB. Let $A_i, i \in K$ be as in Lemma. For every block of D we construct a MEP following the method described in the proof of the lemma above. Let the resultant MEP be named D^* . By Lemma 3.3, every pair of factors belonging to different classes are orthogonal w.r.t. the block factor. Further, since the pair of factors P, Q are mutually orthogonal w.r.t. the block factor in D , it follows that for every $i \in K$, the factors P_i and Q_i are also mutually orthogonal w.r.t. the block factor in D^* . \square

Remark 3.1: If we look at the restriction of D^* to one factor, say P_i , we see that it is nothing but k times repetition of each block of the block design obtained from the restriction of D to the factor P .

4 References

1. Bagchi, S. (2010). Main effect plans orthogonal through the block factor. Technometrics, vol. 52, p. 243-249.
2. Hall, M. (1986). Combinatorial Theory. Wiley-interscience, New York.

3. Morgan, J.P. and Uddin, N. (1996). Optimal blocked main effect plans with nested rows and columns and related designs. *Ann. Stat.* vol. 24, p. 1185-1208.
4. Mukerjee, R., Dey, A. and Chatterjee, K. (2001). Optimal main effect plans with non-orthogonal blocking. *Biometrika*, 89, p. 225-229.
5. Preece, D.A. (1966). Some balanced incomplete block designs for two sets of treatment. *Biometrika* 53, p. 497-506.
6. Rees, D.H. and Preece, D.A. (1999). Perfect Graeco-Latin balanced incomplete block designs. *Disc. Math.* vol.197/198, p. 691-712.
7. Seberry, Jennifer, (1979). A note on orthogonal Graeco-Latin designs. *Ars. Combin.* vol. 8, p. 85-94.
8. Shah, K.R. and Sinha, B.K. (1989). Theory of optimal designs, Lecture notes in Stat., vol. 54, Springer-Verlag, Berlin.
9. Street, D.J. (1981). Graeco-Latin and nested row and column designs. In *Com. Math. VIII, Proc. 8th Austr. Conf. Comb. Math.*, Lecture notes in Math., vol. 884, Springer, Berlin. p. 304-313.
10. Takeuchi, K. (1961). On the optimality of certain type of PBIB designs. *Rep. Stat. Appl. Un. Jpn. Sci. Eng.* vol. 8. p. 140-145.